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Acoustic streaming in an elastico-viscous fluid

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The acoustic streaming of an idealized elastico-viscous fluid near a cylindrical obstacle is considered using the boundary-layer type of approximation. It is shown that this streaming phenomena can be greatly influenced by the presence of elasticity in the fluid.

1. Introduction

It is a well-established fact that when a fluid is set into oscillation, as in the presence of an acoustic wave or an oscillating boundary, steady streaming motions are created. Streaming of this kind had been reported over a century ago by Faraday (1831), and Rayleigh (1883) and Schlichting (1932) were able to give a theoretical explanation of some of these phenomena. More recently, this steady streaming has been considered both theoretically and experimentally by a number of authors (see, for example, Andres & Ingard (1953) and Stuart (1966)). The majority of these investigations have been concerned with either incompressible or compressible viscous fluids.

These motions exert steady stresses on boundaries where the circulation occurs; while these stresses are typically not large they may be significant in continuous removal of loosely adhering surface layers. Also, the unique kind of convection or 'stirring' associated with the acoustic streaming may be especially effective in accelerating certain kinds of rate processes.

In earlier papers (Frater 1964*a*, *b*), it has been shown that, when one considers fluids with marked transient elasticity of shape, the nature of the steady flows produced by oscillations of a boundary can, under certain conditions, be spectacularly different from what is encountered in an inelastic viscous fluid. It is therefore of interest to make a theoretical analysis of some other types of steady streaming looking for differences in observable characteristics from the corresponding cases of viscous flow.

In the present paper, we consider the steady streaming due to the presence of an infinitely long circular cylinder in an infinite expanse of oscillating elastico-viscous fluid, the motion at large distances being transverse to the cylinder axis.

The idealized incompressible elastico-viscous fluid considered has the following equations of state, relating the stress tensor s_{ik} and the rate-of-strain tensor $e_{ik} = \frac{1}{2}(u_{k,i}+u_{i,k})$:

$$s_{ik} = p_{ik} - pg_{ik}, \tag{1}$$

$$p^{ik} + \frac{\lambda_1 \mathfrak{d} p^{ik}}{\mathfrak{d} t} = 2\eta_0 \left(e^{ik} + \frac{\lambda_2 \mathfrak{d} e^{ik}}{\mathfrak{d} t}\right). \tag{2}$$

In these equations, u_i denotes the velocity vector, g_{ik} the metric tensor of a fixed co-ordinate system x^i, p^{ik} the part of the stress tensor related to change of shape of a material element, and p an isotropic pressure; η_0 is a constant having the dimensions of viscosity (which can be identified with the limiting viscosity at vanishingly small constant rate of shear) and λ_1, λ_2 are constants having the dimension of time. The $\mathfrak{d}/\mathfrak{d}t$ is the convected time derivative (Oldroyd 1950) defined thus: if b^{ik} is any contravariant tensor,

$$\frac{\mathfrak{d}b^{ik}}{\mathfrak{d}t} \equiv \frac{\partial b^{ik}}{\partial t} + u^j b^{ik}{}_{,j} + \omega^i_m b^{mk} + \omega^k_m b^{im} - e^i_m b^{mk} - e^k_m b^{im},$$

where $\omega_{ik} = \frac{1}{2}(u_{k,i} - u_{i,k})$ is the vorticity tensor; t is the time.

It has been shown (Oldroyd 1958) that the class of idealized fluids defined by (1) and (2) exhibit many of the observed non-Newtonian features of some polymer solutions and other elastico-viscous fluids, provided the constants η_0, λ_1 and λ_2 are chosen so that

$$\eta_0 > 0 \quad (\lambda_1 > \lambda_2 > 0).$$

In the case of a Newtonian viscous fluid, the present problem was first considered by Rayleigh and later by Schlichting, who used the boundary-layer type of approximation. Schlichting found that the oscillatory potential flow induced a steady-streaming flow which persisted outside the oscillatory boundary layer. More recently, Stuart has shown that, if the Reynolds number of the steady streaming is large, there is an outer boundary layer in which the steady-streaming velocity decays to zero. As a first step in the analysis for an elastico-viscous fluid we follow Schlichting and confine ourselves to a discussion of the steady streaming within the oscillatory boundary layer. Owing to the complexity of (1) and (2) it has not been possible, at the present time, to examine the flow outside the shear-wave layer or to establish that the oscillatory perturbative vorticity does not interact with the potential flow in such a way as to affect the steady streaming. It is felt, however, that the present work is a reasonable first approach to this problem and is sufficient to indicate the effects of elasticity on the streaming phenomenon.

A number of authors have discussed the boundary-layer equations that are applicable in the case of certain elastico-viscous fluids; but these were restricted to steady flows and it appears that no work has yet been done on periodic boundary layers in the fluids characterized by equations of state (1) and (2). Further, apart from the work of Hsu (1967) on the Oldroyd eight constant model, it seems that most discussions have been confined to the Coleman and Noll second-order fluid (see, for example, Beard & Walters (1964)). A detailed examination of Beard & Walters's paper reveals that an error has been made in the order-of-magnitude analysis and, in fact, the steady and non-steady boundary-layer equations for the second-order fluid are identical to those for a Newtonian viscous fluid. The only difference occurs in the expressions for the normal-stresses. By putting certain constants appearing in the equations of state considered by Hsu equal to zero we obtain the result that the steady boundary-layer equations for the fluid characterized by equations (1) and (2) are also identical to those for a Newtonian fluid. The reason for this is that Hsu considered fluids with extremely short memory. It will therefore be of interest to consider what form the periodic boundary-layer equations take when less stringent restrictions are imposed on the relaxation and retardation times of the fluid.

The periodic boundary-layer equations are derived and these are used to calculate the streamfunction for the steady secondary flow in the boundary layer when there is an oscillatory flow parallel to the boundary of magnitude $U(x)\exp(int)$. The convention is adopted that real parts are to be understood whenever complex expressions are quoted for physical quantities.

2. Boundary-layer equations

We consider a non-steady two-dimensional motion with velocity components

$$u = u(x,y,t), \quad v = v(x,y,t), \quad w = 0. \tag{3}$$

The equations of state (2) then reduce to

$$p_{11}+\lambda_1\left(\frac{\partial p_{11}}{\partial t}+u\frac{\partial p_{11}}{\partial x}+v\frac{\partial p_{11}}{\partial y}-2\frac{\partial u}{\partial x}p_{11}-2\frac{\partial u}{\partial y}p_{12}\right)$$
$$= 2\eta_0\left[\frac{\partial u}{\partial x}+\lambda_2\left\{\frac{\partial^2 u}{\partial t\,\partial x}+u\frac{\partial^2 u}{\partial x^2}+v\frac{\partial^2 u}{\partial y\,\partial x}-2\left(\frac{\partial u}{\partial x}\right)^2-\frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}\right], \tag{4}$$

$$p_{22}+\lambda_1\left(\frac{\partial p_{22}}{\partial t}+u\frac{\partial p_{22}}{\partial x}+v\frac{\partial p_{22}}{\partial y}-2\frac{\partial v}{\partial x}p_{12}-2\frac{\partial v}{\partial y}p_{22}\right)$$
$$= 2\eta_0\left[\frac{\partial v}{\partial y}+\lambda_2\left\{\frac{\partial^2 v}{\partial t\,\partial y}+u\frac{\partial^2 v}{\partial x\,\partial y}+v\frac{\partial^2 v}{\partial y^2}-\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)-2\left(\frac{\partial v}{\partial y}\right)^2\right\}\right], \tag{5}$$

$$p_{12}+\lambda_1\left(\frac{\partial p_{12}}{\partial t}+u\frac{\partial p_{12}}{\partial x}+v\frac{\partial p_{12}}{\partial y}-\frac{\partial v}{\partial x}p_{11}-\frac{\partial u}{\partial y}p_{22}\right)$$
$$= \eta_0\left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\lambda_2\left\{\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right.\right.$$
$$\left.\left.+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)-2\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}-2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\}\right]. \tag{6}$$

The stress equations of motion become

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+\frac{\partial p_{11}}{\partial x}+\frac{\partial p_{12}}{\partial y}, \tag{7}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}+\frac{\partial p_{12}}{\partial x}+\frac{\partial p_{22}}{\partial y}, \tag{8}$$

and the incompressibility condition, $e^j_j = 0$, requires

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \tag{9}$$

In these equations, effects of curvature are neglected; x denotes the co-ordinate in the direction of flow, y the co-ordinate normal to the surface.

As equations (4)–(8) stand we are unable to make any boundary-layer type approximations because the orders of magnitude of the stress components inside the boundary layer are unknown. Some progress can be made, however, if we keep in mind Schlichting's analysis of the corresponding problem for a viscous fluid and assume that the amplitude of oscillation is so small compared with the radius of the cylinder that the stresses p_{11}, p_{22}, p_{12}, the velocities u, v and the pressure p can be expanded as:

$$\left.\begin{aligned} p_{11} &= \epsilon p_{11}^{(0)} \exp(int) + \epsilon^2[p_{11}^{(s)} + p_{11}^{(2)} \exp(2\,int)],\\ p_{22} &= \epsilon p_{22}^{(0)} \exp(int) + \epsilon^2[p_{22}^{(s)} + p_{22}^{(2)} \exp(2\,int)],\\ p_{12} &= \epsilon p_{21}^{(0)} \exp(int) + \epsilon^2[p_{21}^{(s)} + p_{21}^{(2)} \exp(2\,int)],\\ u &= \epsilon u_0 \exp(int) + \epsilon^2[u_s + u_2 \exp(2\,int)],\\ v &= \epsilon v_0 \exp(int) + \epsilon^2[v_s + v_2 \exp(2\,int)],\\ p &= \epsilon p_0 \exp(int) + \epsilon^2[p_s + p_2 \exp(2\,int)], \end{aligned}\right\} \tag{10}$$

where ϵ is a dimensionless parameter proportional to the amplitude of oscillation. Here we have anticipated the result that the secondary flow will have a time-independent part plus a part proportional to $\exp(2\,int)$. Stuart (1966) has shown that in the case of a viscous fluid this sort of expansion for the stress and velocity components will be valid near the wall but will not give the correct solution far away from the boundary.

If the expressions (10) are substituted into equations (3)–(9), coefficients of ϵ equated and the stress components eliminated, the following equations are obtained:

$$\rho i n u_0 = -\frac{\partial p_0}{\partial x} + \eta_0 \left(\frac{1+in\lambda_2}{1+in\lambda_1}\right)\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right), \tag{11}$$

$$\rho i n v_0 = -\frac{\partial p_0}{\partial y} + \eta_0 \left(\frac{1+in\lambda_2}{1+in\lambda_1}\right)\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2}\right), \tag{12}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0. \tag{13}$$

To obtain the equations describing the steady secondary flow we substitute the expressions (10) into (3)–(9) and equate coefficients of ϵ^2. If the stress components are then eliminated, we get:

$$\begin{aligned} &\tfrac{1}{4}\rho\left(u_0 \frac{\overline{\partial u_0}}{\partial x} + \bar{u}_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\overline{\partial u_0}}{\partial y} + \bar{v}_0 \frac{\partial u_0}{\partial y}\right) = -\frac{\partial p_s}{\partial x} + \eta_0\left(\frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2}\right)\\ &\quad + \frac{1}{4}\frac{\lambda_2 - \lambda_1}{1+n^2\lambda_1^2}\eta_0\left[2\frac{\partial}{\partial x}\left\{u_0 \frac{\overline{\partial^2 u_0}}{\partial x^2} + \bar{u}_0 \frac{\partial^2 u_0}{\partial x^2} + v_0 \frac{\overline{\partial^2 u_0}}{\partial y\,\partial x} + \bar{v}_0 \frac{\partial^2 u_0}{\partial y\,\partial x}\right.\right.\\ &\qquad\qquad \left. - 4\frac{\partial u_0}{\partial x}\frac{\overline{\partial u_0}}{\partial x} - \frac{\partial u_0}{\partial y}\frac{\overline{\partial v_0}}{\partial x} - \frac{\overline{\partial u_0}}{\partial y}\frac{\partial v_0}{\partial x} - 2\frac{\partial u_0}{\partial y}\frac{\overline{\partial u_0}}{\partial y}\right\}\\ &\quad + \frac{\partial}{\partial y}\left\{u_0 \frac{\partial}{\partial x}\left(\frac{\overline{\partial v_0}}{\partial x} + \frac{\overline{\partial u_0}}{\partial y}\right) + \bar{u}_0 \frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right) + v_0 \frac{\partial}{\partial y}\left(\frac{\overline{\partial v_0}}{\partial x} + \frac{\overline{\partial u_0}}{\partial y}\right) + \bar{v}_0 \frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right)\right.\\ &\qquad\qquad \left.\left. - 2\frac{\partial v_0}{\partial x}\frac{\overline{\partial u_0}}{\partial x} - 2\frac{\overline{\partial v_0}}{\partial x}\frac{\partial u_0}{\partial x} - 2\frac{\partial u_0}{\partial y}\frac{\overline{\partial v_0}}{\partial y} - 2\frac{\overline{\partial u_0}}{\partial y}\frac{\partial v_0}{\partial y}\right\}\right] \end{aligned}$$

$$+\frac{1}{4}\frac{\lambda_2-\lambda_1}{1+n^2\lambda_1^2}in\lambda_1\eta_0\left[2\frac{\partial}{\partial x}\left\{u_0\overline{\frac{\partial^2 u_0}{\partial x^2}}-\bar{u}_0\frac{\partial^2 u_0}{\partial x^2}+v_0\overline{\frac{\partial^2 u_0}{\partial y\,\partial x}}-\bar{v}_0\frac{\partial^2 u_0}{\partial y\,\partial x}-\frac{\partial u_0}{\partial y}\overline{\frac{\partial v_0}{\partial x}}+\overline{\frac{\partial u_0}{\partial y}}\frac{\partial v_0}{\partial x}\right\}\right.$$

$$+\frac{\partial}{\partial y}\left\{u_0\frac{\partial}{\partial x}\left(\overline{\frac{\partial v_0}{\partial x}}+\overline{\frac{\partial u_0}{\partial y}}\right)-\bar{u}_0\frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)+v_0\frac{\partial}{\partial y}\left(\overline{\frac{\partial v_0}{\partial x}}+\overline{\frac{\partial u_0}{\partial y}}\right)-\bar{v}_0\frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right.$$

$$\left.\left.-2\frac{\partial v_0}{\partial x}\overline{\frac{\partial u_0}{\partial x}}+2\overline{\frac{\partial v_0}{\partial x}}\frac{\partial u_0}{\partial x}-2\frac{\partial u_0}{\partial y}\overline{\frac{\partial v_0}{\partial y}}+2\overline{\frac{\partial u_0}{\partial y}}\frac{\partial v_0}{\partial y}\right\}\right], \tag{14}$$

$$\tfrac{1}{4}\rho\left(u_0\overline{\frac{\partial v_0}{\partial x}}+\bar{u}_0\frac{\partial v_0}{\partial x}+v_0\overline{\frac{\partial v_0}{\partial y}}+\bar{v}_0\frac{\partial v_0}{\partial y}\right)=-\frac{\partial p_s}{\partial y}+\eta_0\left(\frac{\partial^2 v_s}{\partial x^2}+\frac{\partial^2 v_s}{\partial y^2}\right)$$

$$+\frac{1}{4}\frac{\lambda_2-\lambda_1}{1+n^2\lambda_1^2}\eta_0\left[\frac{\partial}{\partial x}\left\{u_0\frac{\partial}{\partial x}\left(\overline{\frac{\partial v_0}{\partial x}}+\overline{\frac{\partial u_0}{\partial y}}\right)+\bar{u}_0\frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)+v_0\frac{\partial}{\partial y}\left(\overline{\frac{\partial v_0}{\partial x}}+\overline{\frac{\partial u_0}{\partial y}}\right)\right.\right.$$

$$\left.+\bar{v}_0\frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)-2\frac{\partial v_0}{\partial x}\overline{\frac{\partial u_0}{\partial x}}-2\overline{\frac{\partial v_0}{\partial x}}\frac{\partial u_0}{\partial x}-2\frac{\partial u_0}{\partial y}\overline{\frac{\partial v_0}{\partial y}}-2\overline{\frac{\partial u_0}{\partial y}}\frac{\partial v_0}{\partial y}\right\}$$

$$\left.+2\frac{\partial}{\partial y}\left\{u_0\overline{\frac{\partial^2 v_0}{\partial x\,\partial y}}+\bar{u}_0\frac{\partial^2 v_0}{\partial x\,\partial y}+v_0\overline{\frac{\partial^2 v_0}{\partial y^2}}+\bar{v}_0\frac{\partial^2 v_0}{\partial y^2}-2\frac{\partial v_0}{\partial x}\overline{\frac{\partial v_0}{\partial x}}-\frac{\partial v_0}{\partial x}\overline{\frac{\partial u_0}{\partial y}}-\overline{\frac{\partial v_0}{\partial x}}\frac{\partial u_0}{\partial y}-4\frac{\partial v_0}{\partial y}\overline{\frac{\partial v_0}{\partial y}}\right\}\right]$$

$$+\frac{1}{4}\frac{\lambda_2-\lambda_1}{1+n^2\lambda_1^2}in\lambda_1\eta_0\left[\frac{\partial}{\partial x}\left\{u_0\frac{\partial}{\partial x}\left(\overline{\frac{\partial v_0}{\partial x}}+\overline{\frac{\partial u_0}{\partial y}}\right)-\bar{u}_0\frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)+v_0\frac{\partial}{\partial y}\left(\overline{\frac{\partial v_0}{\partial x}}+\overline{\frac{\partial u_0}{\partial y}}\right)\right.\right.$$

$$\left.-\bar{v}_0\frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)-2\frac{\partial v_0}{\partial x}\overline{\frac{\partial u_0}{\partial x}}+2\overline{\frac{\partial v_0}{\partial x}}\frac{\partial u_0}{\partial x}-2\frac{\partial u_0}{\partial y}\overline{\frac{\partial v_0}{\partial y}}+2\overline{\frac{\partial u_0}{\partial y}}\frac{\partial v_0}{\partial y}\right\}$$

$$\left.+2\frac{\partial}{\partial y}\left\{u_0\overline{\frac{\partial^2 v_0}{\partial x\,\partial y}}-\bar{u}_0\frac{\partial^2 v_0}{\partial x\,\partial y}+v_0\overline{\frac{\partial^2 v_0}{\partial y^2}}-\bar{v}_0\frac{\partial^2 v_0}{\partial y^2}-\frac{\partial v_0}{\partial x}\overline{\frac{\partial u_0}{\partial y}}+\overline{\frac{\partial v_0}{\partial x}}\frac{\partial u_0}{\partial y}\right\}\right], \tag{15}$$

$$\frac{\partial u_s}{\partial x}+\frac{\partial v_s}{\partial y}=0. \tag{16}$$

We take $U(x)\exp(int)$ to be the velocity at the edge of the boundary layer. Then if U_0 is a typical velocity, d a typical length and n^{-1} a typical time associated with the flow we may define the dimensionless parameter ϵ and another dimensionless parameter δ by the following equations,

$$\epsilon=\frac{U_0}{nd},\quad \delta=\frac{\eta_0}{\rho n d^2}. \tag{17}$$

We have already assumed that ϵ is small compared with unity and for the case we now consider we take δ to be small compared with ϵ. Under these conditions we find $\partial p_0/\partial y=0$, $\partial p_s/\partial y=0$ to a first approximation so that $\partial p_0/\partial x$ and $\partial p_s/\partial x$ may be replaced by their values at the edge of the boundary layer, namely

$$-\frac{\partial p_0}{\partial x}=\rho inU, \tag{18}$$

$$-\frac{\partial p_s}{\partial x}=\tfrac{1}{2}\rho U\frac{dU}{dx}. \tag{19}$$

Here we have assumed that the flow external to the boundary layer is that of an inviscid fluid. This will be so provided $n\lambda_1$ and $n\lambda_2$ are small compared with ϵ/δ. Substituting for $\partial p_0/\partial x$ and $\partial p_s/\partial x$ and neglecting terms of order δ, equations (11) and (14) reduce to, respectively,

$$\rho inu_0 = \rho inU + \eta_0 \left(\frac{1+in\lambda_2}{1+in\lambda_1}\right) \frac{\partial^2 u_0}{\partial y^2}, \tag{20}$$

$$\begin{aligned}\tfrac{1}{4}\rho\left(u_0 \overline{\frac{\partial u_0}{\partial x}} + \bar{u}_0 \frac{\partial u_0}{\partial x} + v_0 \overline{\frac{\partial u_0}{\partial y}} + \bar{v}_0 \frac{\partial u_0}{\partial y}\right) &= \tfrac{1}{2}\rho U \frac{dU}{dx} + \eta_0 \frac{\partial^2 u_s}{\partial y^2} \\ &+ \frac{1}{4}\frac{\lambda_2-\lambda_1}{1+n^2\lambda_1^2}\eta_0\left(u_0 \overline{\frac{\partial^3 u_0}{\partial x\,\partial y^2}} + \bar{u}_0 \frac{\partial^3 u_0}{\partial x\,\partial y^2} + v_0 \overline{\frac{\partial^3 u_0}{\partial y^3}} + \bar{v}_0 \frac{\partial^3 u_0}{\partial y^3}\right. \\ &\left.+ \frac{\partial u_0}{\partial x}\overline{\frac{\partial^2 u_0}{\partial y^2}} + \overline{\frac{\partial u_0}{\partial x}}\frac{\partial^2 u_0}{\partial y^2} - \frac{\partial u_0}{\partial y}\overline{\frac{\partial^2 u_0}{\partial x\,\partial y}} - \overline{\frac{\partial u_0}{\partial y}}\frac{\partial^2 u_0}{\partial x\,\partial y}\right) \\ &+ \frac{1}{4}\frac{\lambda_2-\lambda_1}{1+n^2\lambda_1^2} in\lambda_1\eta_0\left(u_0 \overline{\frac{\partial^3 u_0}{\partial x\,\partial y^2}} - \bar{u}_0 \frac{\partial^3 u_0}{\partial x\,\partial y^2} + v_0 \overline{\frac{\partial^3 u_0}{\partial y^3}} - \bar{v}_0 \frac{\partial^3 u_0}{\partial y^3}\right. \\ &\left.- 3\frac{\partial u_0}{\partial x}\overline{\frac{\partial^2 u_0}{\partial y^2}} + 3\overline{\frac{\partial u_0}{\partial x}}\frac{\partial^2 u_0}{\partial y^2} + 3\frac{\partial u_0}{\partial y}\overline{\frac{\partial^2 u_0}{\partial x\,\partial y}} - 3\overline{\frac{\partial u_0}{\partial y}}\frac{\partial^2 u_0}{\partial x\,\partial y}\right).\end{aligned} \tag{21}$$

The appropriate boundary conditions are:

$$\left.\begin{aligned} u_0 = 0, \quad v_0 = 0 \quad &\text{on} \quad y = 0, \\ u_0 \to U \quad \text{as} \quad &y \to \infty, \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} u_s = 0, \quad v_s = 0 \quad &\text{on} \quad y = 0, \\ u_s \to U_s \quad \text{as} \quad &y \to \infty. \end{aligned}\right\} \tag{23}$$

Here we have anticipated the result that within the framework of the present theory it is not possible to satisfy both the boundary conditions (23) at the wall and the condition $u_s \to 0$ as $y \to \infty$. Instead the condition at infinity is relaxed to 'u_s remains finite as $y \to \infty$'. For a complete discussion of this point, the reader is referred to Stuart (1966) and Riley (1965).

3. Solution of the equations

From equation (13), it is possible to define a function $\psi_0(x, y)$ such that

$$u_0 = -\frac{\partial \psi_0}{\partial y}, \quad v_0 = \frac{\partial \psi_0}{\partial x}. \tag{24}$$

On writing
$$\eta = \left(\frac{n\rho}{2\eta_0}\right)^{\frac{1}{2}} y, \quad \psi_0 = \left(\frac{2\eta_0}{n\rho}\right)^{\frac{1}{2}} U(x)\,\zeta_1(\eta),$$

and using (24), equation (20) reduces to the following ordinary differential equation

$$\frac{d^3\zeta_1}{d\eta^3} = \alpha^2\left(\frac{d\zeta_1}{d\eta} + 1\right), \tag{25}$$

where
$$\alpha^2 = 2i\left(\frac{1+in\lambda_1}{1+in\lambda_2}\right). \tag{26}$$

The associated boundary conditions are

$$\left.\begin{aligned} \frac{d\zeta_1}{d\eta} &= 0, \quad \zeta_1 = 0 \quad \text{on} \quad \eta = 0, \\ \frac{d\zeta_1}{d\eta} &= -1 \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{27}$$

It follows that
$$\zeta_1 = (1 - e^{-\alpha\eta})/\alpha - \eta. \tag{28}$$

The equation of continuity (16) is the condition for the existence of a stream-function $\psi_s(x, y)$ such that

$$u_s = -\frac{\partial \psi_s}{\partial y}, \quad v_s = \frac{\partial \psi_s}{\partial x}. \tag{29}$$

On writing
$$\psi_s = \left(\frac{2\eta_0}{\rho n^3}\right)^{\frac{1}{2}} U(x) \frac{dU(x)}{dx} \zeta_2(\eta), \tag{30}$$

and using (29) we find that equation (21) reduces to

$$\begin{aligned} \zeta_2''' = {} & 1 - \zeta_1' \bar{\zeta}_1' + \tfrac{1}{2}(\zeta_1 \bar{\zeta}_1'' + \bar{\zeta}_1 \zeta_1'') \\ & + \frac{1}{4} \frac{\lambda_2 - \lambda_1}{1 + n^2\lambda_1^2} n \,(2\zeta_1' \bar{\zeta}_1''' + 2\bar{\zeta}_1' \zeta_1''' - \zeta_1 \bar{\zeta}_1^{\text{iv}} - \bar{\zeta}_1 \zeta_1^{\text{iv}} - 2\zeta_1'' \bar{\zeta}_1'') \\ & + \frac{1}{4} \frac{\lambda_2 - \lambda_1}{1 + n^2\lambda_1^2} in^2\lambda_1(-2\zeta_1' \bar{\zeta}_1''' + 2\bar{\zeta}_1' \zeta_1''' - \zeta_1 \bar{\zeta}_1^{\text{iv}} + \bar{\zeta}_1 \zeta_1^{\text{iv}}), \end{aligned} \tag{31}$$

where a prime denotes differentiation with respect to η. This equation is to be solved subject to the conditions

$$\left.\begin{aligned} & \zeta_2' = 0, \quad \zeta_2 = 0 \quad \text{on} \quad \eta = 0, \\ & \frac{U(x)}{n} \frac{dU(x)}{dx} \zeta_2' = -U_s \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{32}$$

Substituting for ζ_1 and performing the necessary integrations we find

$$\begin{aligned} \zeta_2 = L_0 e^{-2a\eta} + L_1 e^{-a\eta} \cos b\eta + L_2 e^{-a\eta} \sin b\eta + L_3 \eta e^{-a\eta} \cos b\eta \\ + L_4 \eta e^{-a\eta} \sin b\eta + \tfrac{1}{2}A\eta^2 + B\eta + C, \end{aligned} \tag{33}$$

where A, B and C are constants of integration, a and b are the real and imaginary parts of α and L_i ($i = 0, \ldots, 4$) are functions of a and b which are too complicated to be given here. Using the boundary conditions (31) we find immediately $A = 0$, $B = 2aL_0 + aL_1 - bL_2 - L_3$ and $C = -(L_0 + L_1)$. It follows that $U_s(x)$ is given by

$$\begin{aligned} U_s(x) &= -\frac{U(x)}{n} \frac{dU(x)}{dx} (2aL_0 + aL_1 - bL_2 - L_3), \\ &= -\mu \frac{U(x)}{n} \frac{dU(x)}{dx}, \quad \text{say.} \end{aligned} \tag{34}$$

We are now in a position to apply the foregoing results to the case of oscillating flow past a cylinder. For a cylinder we have

$$U(x) = 2U_0 \sin(x/d), \tag{35}$$

where x is measured along the boundary from a stagnation point. If we define the angle ϕ to be x/d, we then have

$$U(x) = 2U_0 \sin\phi. \tag{36}$$

It follows that
$$\frac{dU(x)}{dx} = \frac{2U_0}{d}\cos\frac{x}{d} = \frac{2U_0}{d}\cos\phi. \tag{37}$$

Substituting these values into (30) we find that the streamfunction for the steady part of the secondary flow is

$$\psi_s = 2\left(\frac{2\eta_0}{\rho n^3}\right)^{\frac{1}{2}} \frac{U_0^2}{d}\zeta_2(\eta)\sin 2\phi, \tag{38}$$

where $\zeta_2(\eta)$ is given by (33).

λ_1/λ_2	η^*	
1	1·8	$n\lambda_2 = 0{\cdot}03$
3	1·9	
5	2·0	
7	2·1	
9	2·2	
1	1·8	$n\lambda_2 = 0{\cdot}06$
3	2·0	
5	2·2	
7	2·6	
9	3·1	
1	1·8	$n\lambda_2 = 0{\cdot}09$
3	2·1	
5	2·6	
7	3·5	
9	10·9	

TABLE 1

The general forms of the functions ψ_s and U_s are not sufficiently simple to permit a discussion of the flow pattern without reference to particular cases. Numerical calculations have been performed for the cases $\lambda_1/\lambda_2 = 3$, 5, 7, 9 and the Newtonian case ($\lambda_1/\lambda_2 = 1$), under each of the conditions $n\lambda_2 = 0{\cdot}03, 0{\cdot}06, 0{\cdot}09$.

In each case, it is found that there are two regions of flow (for a sketch of the streamlines in the Newtonian case, see Andres & Ingard (1953)). In the outer region the fluid flows away from the cylinder in the direction of oscillation and towards it in the transverse direction. In the inner region the streamlines are closed and the fluid flows towards the cylinder in the direction of oscillation and away from it in the transverse direction. Table 1 gives details of the variation of the thickness of the inner region (η^* say) with $n\lambda_2$ for each of the above cases. These results show that for $n\lambda_2 = 0{\cdot}03$ the thickness of the inner region is only slightly affected by the presence of elasticity in the fluid but for higher values of $n\lambda_2$ the effect of the elasticity is to increase this thickness; the greater the value λ_1/λ_2 the greater the increase. Quite substantial increases being obtained in certain cases.

Details of the function μ are given in table 2. In the case of a Newtonian fluid, the function μ is independent of frequency and is given by $\mu = 0{\cdot}75$. The results show that μ, which is proportional to the steady second-order velocity just outside the boundary layer, can both exceed and be less than the corresponding value for the viscous fluid, depending on the magnitudes of the dimensionless groups $n\lambda_2$ and λ_1/λ_2. In particular, when $n\lambda_2 = 0{\cdot}09$, this steady drift velocity in the case $\lambda_1/\lambda_2 = 9$ is only about $\frac{1}{8}$th the corresponding value for the viscous fluid.

λ_1/λ_2	μ	
3	0·773	$n\lambda_2 = 0{\cdot}03$
5	0·785	
7	0·782	
9	0·763	
3	0·780	$n\lambda_2 = 0{\cdot}06$
5	0·754	
7	0·662	
9	0·510	
3	0·768	$n\lambda_2 = 0{\cdot}09$
5	0·652	
7	0·410	
9	0·097	

TABLE 2

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